

Heun-type solutions for Schwarzschild metric with electromagnetic fields

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Abstract

We find confluent Heun solutions to the radial equations of two Halilsoy-Badawi metrics. For the first metric, we studied the radial part of the massless Dirac equation and for the second case, we studied the radial part of the massless Klein-Gordon equation.

1 Introduction

Heun functions [1, 2] seem to be still a novelty among theoretical physicists although they were introduced nearly 130 years ago. After the centennial conference which took place in 1989 and the papers presented in this conference were published in a book [2], there was an explosion of papers in this field [3]. Many equations whose exact solutions were not known turn out to have solutions in this set. After referring to people [4–6] who tried to show whether the celebrated “Teukolsky Master Equation” [7] was one of the confluent forms of the Heun equation, Batic and Schmidt [8] showed that the “Teukolsky master Equation” could be transformed in any relevant D type metric into a Heun form. Although the Heun equation and its confluent forms are much better known today in the theoretical physics community and included in some mathematical packages, we still find some authors who do not identify the equations they find properly.

Here we give two examples of metrics which yield confluent Heun solutions in their wave equations. In the first example the massless Dirac equation and in the second, the massless Klein-Gordon equation are studied.

2 Dirac equation

In a very interesting paper [9] A. Al-Badawi, M.Q.Owaidat study the Dirac equation when the Schwarzschild mass is coupled to a stationary electromagnetic field. For the metric they use a solution found by one of these authors with a collaborator [10]. This solution is a superposition of the Schwarzschild [11] solution to an external, stationary electromagnetic Bertotti-Robinson solution [12, 13]. They use the Newman-Penrose formalism [14] to separate the Dirac equation into radial and angular (θ) parts after assuming a periodic solution for the variables t and ϕ which are along Killing directions. They can solve the angular equation in terms of the associated Legendre functions. For the radial equation, they use the WKB approximation and write their solutions in terms of exponentials.

For the metric

$$ds^2 = \frac{r^2 - 2Mr}{r^2 f(r)} [dt - Mq(1 + a^2) \cos \theta d\phi]^2 - \frac{r^2 f(r)}{r^2 - 2Mr} dr^2 - r^2 f(r) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

in the massless case, the radial equations are of the form:

$$\frac{dT_1}{dr} + i \frac{kR^2}{H^2 R^2} T_1 = \frac{\lambda}{RH} T_2, \quad (2)$$

$$\frac{dT_2}{dr} - i \frac{kR^2}{H^2 R^2} T_2 = \frac{\lambda}{RH} T_1. \quad (3)$$

Here λ is the eigenvalue and $R^2 = r^2 f(r)$ where

$$r^2 f(r) = \frac{1}{2} r (r - 2M) [p(1 + a^2) + a^2 - 1] + 2Mar + M^2 [p(1 + a^2) - 2a], \quad (4)$$

and

$$H^2 = \frac{r^2 - 2Mr}{r^2 f(r)}. \quad (5)$$

M is the mass of the Schwarzschild mass, p is the twisting parameter of the external electromagnetic field and a is the interpolation parameter between two metrics used.

The exact solutions for the radial equations turn out to a confluent Heun functions multiplied by exponentials and powers of the finite regular singular points of the radial equation. This is no surprise, since as Philipp and Perlick point out [15], Leaver [5,16] and Fiziev [17–19] showed that the solution of a particle in the Schwarzschild background results in confluent Heun type solutions. In this respect, it looks like the solutions of the Eguchi-Hanson instanton [20] trivially extended to five dimensions [21]. In this case, the angular equation has a hypergeometric function solution, whereas the radial equation's solution is again a confluent Heun function [1,2].

In their paper [9] we could not find the explicit expressions for the frequency ω aside from an integral, to check the behavior of the incoming and outgoing waves as well as the expressions for the transition and reflection coefficients. The authors end their paper by studying the plots of the effective potentials for different values of the parameters in the effective potential.

One can show that the radial equation can indeed be solved in terms of confluent Heun functions. This solution has regular singularities at $r = 0$ and $r = 2M$ and an irregular singularity at infinity. We first study the solution for $0 < r < 2M$, assuming that the given metric is also valid here. We are aware of the fact that the Schwarzschild metric is only valid outside the event horizon. We do this just to compare our solution with that given in [9]. We only take the first solution which is analytic around the singularity at $r = 0$. It reads as:

$$\begin{aligned} T_1(r) = & e^{i/2((p+1)a^2+p-1)rk} r^{i/2k(a^2p-2a+p)M} (2M-r)^{1/2+i/2k(a^2p+2a+p)M} \times \\ & H_C \left(2iM((p+1)a^2+p-1)k, -1/2+ik(a^2p-2a+p)M, \right. \\ & 1/2+ik(a^2p+2a+p)M, (4Mak+i)M((p+1)a^2+p-1)k, \\ & 1/2k^2M^2(a^2+1)^2p^2 \\ & -1/2M(a^2+1)(-2Ma^2k+4Mak+2kM+i)kp-2k^2a^3M^2 \\ & \left. -1/2Mk(-4kM+i)a^2+M(2kM+i)ka+i/2kM-\lambda^2+3/8, \frac{r}{2M} \right), \end{aligned} \quad (6)$$

where H_C denotes the confluent Heun function. We note that although this solution is analytic around $r = 0$, it is not analytic around $r = 2M$. Here the parameters are defined in Reference [9].

The standard form of the confluent Heun equation is given as [22,23]

$$\frac{d^2 H_C}{dz^2} + \left(\alpha + \frac{\gamma+1}{z-1} + \frac{\beta+1}{z} \right) \frac{dH_C}{dz} + \left(\frac{\mu}{z} + \frac{\nu}{z-1} \right) H_C = 0, \quad (7)$$

with solution $H_C(\alpha, \beta, \gamma, \delta, \eta, z)$, and the parameters have the relations

$$\delta = \mu + \nu - \alpha \left(\frac{\beta + \gamma + 2}{2} \right), \quad (8)$$

$$\eta = \frac{\alpha(\beta+1)}{2} - \mu - \frac{\beta + \gamma + \beta\gamma}{2}. \quad (9)$$

Since we are interested in the region $r > 2M$, outside the event horizon, the solution we gave above does not suit our purposes if we want to investigate the behavior of the wave for $r > 2M$. To find a solution to suit our purpose, we have to transform to the variable $u = r - 2M$. This will give us one solution which is analytic around $r = 2M$. This solution may not be analytic around $r = 0$. Since we are not interested in the region

$0 < r < 2M$, this will not cause any problems. Here we again take the solution which is analytic around $u = 0$ ($r = 2M$), namely

$$\begin{aligned}
T_1(u) = & e^{-i/2uk((p+1)a^2+p-1)} (u+2M)^{i/2k(a^2p-2a+p)M} u^{-i/2(a^2p+2a+p)kM} \times \\
& H_C \left(2iM((p+1)a^2+p-1)k, -1/2 - i(a^2p+2a+p)kM, \right. \\
& -1/2 + ik(a^2p-2a+p)M, -M((p+1)a^2+p-1)(4kMa+i)k, \\
& 1/2 M^2 k^2 (a^2+1)^2 p^2 + 1/2 M(a^2+1)(2Ma^2k+4kMa-2Mk+i)kp \\
& \left. + 2k^2 a^3 M^2 + 1/2 M(4Mk+i)ka^2 + Mk(-2Mk+i)a - i/2Mk - \lambda^2 + 3/8, -\frac{u}{2M} \right). \tag{10}
\end{aligned}$$

Using $p = 10$, $k = 0.2$, $a = 0.1$, $\lambda = 0.7$ and $M = 5$, we give the plots of this solution for $0 < r < 2M$ and $u > 0$ in Figure 1 and Figure 2, respectively.

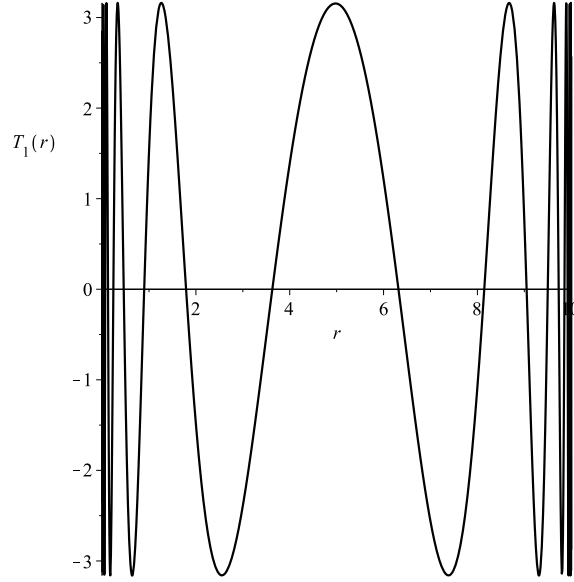


Figure 1: Solution between $0 < r < 2M$

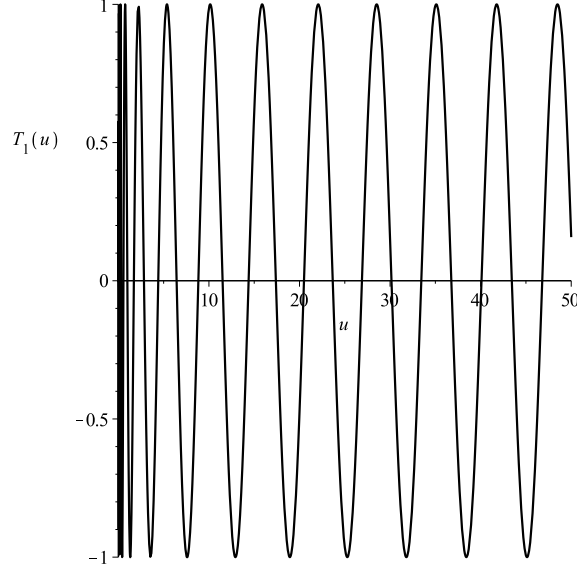


Figure 2: Solution for $u > 0$

Polynomial solutions can be given for the confluent Heun equation under some conditions [23, 24]. The identity $\mu + \nu = -N\alpha$, N being the degree of the polynomial solution, should be satisfied along with a vanishing determinant. However, this identity is not useful in this case as $\mu + \nu = 0$.

Solution around infinity

Solution of the confluent Heun equation around the irregular singularity at infinity can be given by the Thomé solution as [25]

$$\lim_{z \rightarrow \infty} U(z) \sim e^{\pm i\omega z} z^{\mp i\eta - (B_2/2)}, \quad (11)$$

where the confluent Heun equation is written in the form

$$z(z-1)\frac{d^2U}{dz^2} + (B_1 + B_2z)\frac{dU}{dz} + [B_3 - 2\eta\omega(z-1) + \omega^2z(z-1)]U = 0, \quad (12)$$

for $\omega \neq 0$ and all other parameters are constants. This form is called the generalized spheroidal wave equation and finding the correspondence with the general form given by the equation (7) needs some algebra as the solutions are not in the same form (*i.e.* we need to define $V(z) = e^{i\omega z}U(z)$ first and then proceed with the solution). Studying the solution for our case, we find

$$\lim_{u \rightarrow \infty} T_1(u) \sim e^{(2i[(p+1)a^2 + p-1])Mku}, \quad (13)$$

as the solution around infinity. There is another solution which goes as $u^{4ikMa-1}$.

3 Klein-Gordon equation

In another recent paper by Al-Badawi, Dirac equation is studied in Schwarzschild black hole immersed in an electromagnetic universe with charge coupling [26]. The metric is

$$ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{r^2}{\Delta} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (14)$$

where $\Delta = r^2 - 2Mr + M^2(1 - a^2)$ [27]. Here M is the S mass coupled to an external electromagnetic field and a ($0 < a \leq 1$) is the external parameter.

We studied the massless Klein-Gordon equation in the same metric, namely

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) = 0, \quad (15)$$

in this background. This equation can be separated into the radial and angular parts with the Ansatz

$$\Phi = e^{-i\omega t} e^{in\phi} F(r)S(\theta). \quad (16)$$

The angular part is in the form of the associated Legendre equation and the radial part can be solved in terms of confluent Heun functions. We change our parameter r to $u = r - r_1$, r_1 being the event horizon in order to study the behavior outside the event horizon. We note that the event horizon is located at $r_1 = M(1 + a)$ and the inner horizon is located at $r_2 = M(1 - a)$. The radial solution is

$$F(u) = e^{-i\omega u} u^{\frac{ir_1^2\omega}{r_1-r_2}} (u + r_1 - r_2)^{\frac{ir_2^2\omega}{r_1-r_2}} \times H_C \left(2i\omega(r_1 - r_2), \frac{2ir_1^2\omega}{r_1 - r_2}, \frac{2ir_2^2\omega}{r_1 - r_2}, (-2r_1^2 + 2r_2^2)\omega^2, \frac{2r_1^4\omega^2 - 4r_1^3\omega^2 r_2 - \lambda r_1^2 + 2r_1 r_2 \lambda - \lambda r_2^2}{(r_1 - r_2)^2}, -\frac{u}{r_1 - r_2} \right). \quad (17)$$

4 Conclusion

Here we studied two different metrics given by [10] and [27]. In the first case we studied the Dirac equation given in [9] and found that the radial solution can be expressed in terms of confluent Heun functions. We found the same structure in the second metric case [27] for the Klein-Gordon equation.

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